**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH2050A Mathematical Analysis I (Fall 2022)**
**Suggested Solution of Homework 5**

(1) (a) Fix $\epsilon > 0$. Take $\delta = \frac{\epsilon}{4a^3}$. If $|x-y| < \delta$, then

$$|x^4 - y^4| = |x^2 + y^2||x+y||x-y| < 4a^3\delta = \epsilon.$$

Hence, $f$ is uniformly continuous on $[0, a]$.

(b) Take $x_n = (n+1)^{1/4}$, $y_n = n^{1/4}$. Then for any $n \in \mathbb{N}$

$$|x_n^4 - y_n^4| = 1,$$

and

$$|x_n - y_n| = \frac{1}{((n+1)^{1/2} + n^{1/2})((n+1)^{1/4} + n^{1/4})} \to 0$$

as $n \to \infty$. Hence, $f$ is not uniformly continuous on $[0, \infty)$.

(2) (a) Take $x_n = \frac{1}{n^3}$. Then

$$\frac{|x_n^{1/3} - 0|}{x_n - 0} = \frac{1/n}{1/n^3} = n^2 \to \infty$$

as $n \to \infty$. Hence, $f$ is not Lipschitz on $[0, 1]$.

(b) Fix $\epsilon > 0$. Take $\delta = \frac{\epsilon^3}{2}$. If $|x-y| < \delta$, then

$$\begin{aligned}
|x^{1/3} - y^{1/3}| &\leq (|x-y| + |3x^{1/3}y^{1/3}||x^{1/3} - y^{1/3}|)^{1/3} \\
&\leq (|x-y| + |x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}||x^{1/3} - y^{1/3}|)^{1/3} \\
&= (2|x-y|)^{1/3} \\
&< (2\delta)^{1/3} \\
&= \epsilon.
\end{aligned}$$

Hence, $f$ is uniformly continuous on $[0, 1]$.

(3) (a) Since $f$ is continuous on $[0, k]$, $f$ is uniformly continuous on $[0, k]$. Thus there exists $M > 0$ such that $|f(x)| < M$ for any $x \in [0, k]$. Take $\tilde{L} = \max\{L, M\}$. Then $|f(x)| < \tilde{L}$ for any $x \in [0, \infty)$.

(b) Fix $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = \alpha$, there exists $M > 0$ such that for any $x > M$, $|f(x) - \alpha| < \frac{\epsilon}{2}$. If $x, y > M$, then $|f(x) - f(y)| \leq |f(x) - \alpha| + |f(y) - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. On the other hand, since $f$ is continuous on $[0, M+1]$, $f$ is uniformly continuous on $[0, M+1]$. Thus there exists $\delta > 0$ such that for any $x, y \in [0, M+1]$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Take $\tilde{\delta} = \min\{\delta, 1\}$. For any $x, y \in [0, M+1]$, if $|x-y| < \tilde{\delta}$, then either $x, y \in [0.M+1]$ or $x, y \in [M, \infty)$. If $x, y \in [M, \infty)$, then $|f(x) - f(y)| < \epsilon$. If $x, y \in [0.M+1]$, since $|x-y| < \tilde{\delta} \leq \delta$, $|f(x) - f(y)| < \epsilon$.

Hence, $f$ is uniformly continuous on $[0, \infty)$.

(4) Suppose there exists $x, y \in [0, 1]$ such that $f(x) \neq f(y)$. Without loss of generality, assume $f(x) < f(y)$. By density of $\mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $f(x) < q < f(y)$. Since $f$ is continuous, by Intermediate Value Theorem, there exists $x < z < y$ such that $f(z) = q$. Contradiction arises!